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THE STABILITY OF MECHANICAL SYSTEMS WITH POSITIONAL NON-CONSERVATIVE FORCES†

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A classical problem is discussed, namely, the influence of the structure of the applied forces on the stability of the equilibrium position of an autonomous mechanical system. Several propositions extending the Thomson–Tate–Chetayev theorems to systems with non-conservative positional forces are proved. © 2003 Elsevier Ltd. All rights reserved.

The problem in question has been considered by many scholars. The most complete and general theorems have been proved for the case in which the positional forces are conservative [1]. Subsequent interest in generalizations was stimulated by several critical practical problems in which the presence of non-conservative positional forces leads to catastrophic outcomes [2]. Some general results relating to the stability of systems with such forces have been presented in [3].

1. FORMULATION OF THE PROBLEM

The equations of motion of an autonomous system with generalized coordinates $\mathbf{q} \in R^n$ may be represented in the neighbourhood of the equilibrium position $\mathbf{q} = \mathbf{0}$ in the form

$$M\ddot{\mathbf{q}} + (D+G)\dot{\mathbf{q}} + (K+N)\mathbf{q} = \mathbf{Q}_2 \tag{1.1}$$

where $\mathbf{Q}_2$ denotes all the non-linear terms, and the matrices M, D, K, G, and N are constants; the first three of the latter, which describe the mass distribution, dissipation and potential forces, are symmetric, while the matrices of gyroscopic and circulation forces are skew-symmetric. We shall assume that $D > 0$, and then, in a typical case, the question of whether the equilibrium position is stable may be settled by considering the linear approximation

$$M\ddot{\mathbf{q}} + (D+G)\dot{\mathbf{q}} + (K+N)\mathbf{q} = 0 \tag{1.2}$$

If $N = 0$, the question of the stability of system (1.1) may be settled on the basis of the Thomson–Tate–Chetayev theorems, based on an analysis of the matrix K: if it is positive, the system is asymptotically stable; but if it has at least one negative eigenvalue, it is unstable. Both these conclusions are valid irrespective of the form of the matrices $D > 0$, and G; this observation is of practical importance, since it is usually impossible to make precise allowance for dissipative forces.

It was proposed in [3] to extend the above to the case $N \neq 0$ by applying a coordinate transformation

$$\mathbf{q} = L(t)\boldsymbol{\xi} \tag{1.3}$$

which yields, in the new coordinates, a system analogous to (1.2) but without circulation forces, while the matrix of dissipative forces has the same form. One can then apply the above-mentioned theorems. However, the existence of a normalizing transformation (1.3) has been proved only for certain additional assumptions about the form of the matrices D and N.

There are also several other sufficient conditions for system (1.2) to be asymptotically stable, involving various restrictions on the matrix D [3].

In this paper the matrix $D > 0$ is assumed to be arbitrary. The main result is stated below.

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Proposition 1. Let the matrices K and N be such that system (1.2), in the absence of forces that depend on velocities (i.e., in the case when $D = G = 0$), is stable. Then, if there are dissipative forces with an arbitrary matrix $D > 0$, the system admits of gyrostabilization. In order words, a skew-symmetric matrix G, depending on D, exists, such that the trivial equilibrium of position of system (1.2) is asymptotically stable.

Remark. In the case when $D = G = 0$, system (1.2) is reciprocal, and its stability (instability) is achieved only provided all the roots of the characteristic equation

$$\det(M\lambda^2 + K + N) = 0$$

are pure imaginary and the elementary divisors are simple. In other words, the matrix $M^{-1}(K + N)$ is positive and diagonal in a certain basis.

2. PROOF OF PROPOSITION 1

As is well known [4], we may assume, without loss of generality, that the matrix M in system (1.2) is the identity matrix. This simplification may be effected by applying a transformation $\mathbf{q} \to \mathbf{z}$, $\mathbf{q} = \Lambda\mathbf{z}$, and simultaneously multiplying Eq. (1.2) on the left by Λ^T, where Λ is some non-singular matrix. As a result we obtain the system

$$\ddot{\mathbf{z}} + (D' + G')\dot{\mathbf{z}} + (K' + N')\mathbf{z} = 0 \tag{2.1}$$

where $D' = \Lambda^T D\Lambda$ and so on. It is important that these transformations preserve the structure of the system: the matrices D' and K' are symmetric, G' and N' are skew-symmetric, and $D > 0 \Leftrightarrow D' > 0$.

The next step consists of applying the transformation

$$\mathbf{z} = \Gamma\mathbf{x}, \quad \Gamma = AA^T \tag{2.2}$$

where A is the transformation matrix from the natural basis in R^n to a basis consisting of eigenvectors of the matrix $K' + N'$ (the existence of such a basis follows from the assumptions of the proposition; see the Remark above).

Equations (2.1) become

$$\Gamma\ddot{\mathbf{x}} + U\dot{\mathbf{x}} + V\mathbf{x} = 0, \quad U = (D' + G')\Gamma, \quad V = (K' + N')\Gamma \tag{2.3}$$

As follows from the definition of the matrix Γ, it is symmetric. We shall show that the matrix V is also symmetric. Indeed,

$$\begin{aligned} V^T &= \Gamma^T(K' + N')^T = AA^T(K' + N')^T = AA^T(K' + N')^T(A^{-1})^T A^T = \\ &= A(A^{-1}(K' + N')A)^T = AA^{-1}(K' + N')AA^T = (K' + N')\Gamma = V \end{aligned}$$

When deriving this formula it was taken into consideration that the matrix $A^{-1}(K' + N')A$ is diagonal and therefore symmetric.

The matrices Γ and V is system (2.3) are symmetric and positive (since by assumption the system is stable under certain positional forces). For the Thomson–Tate–Chetayev theorem to be applicable to this system, it will suffice to verify that the symmetric part U_s of the matrix U is positive. The verification is not trivial, because the transformation (2.2) does not preserve the structure of the forces, and the matrix U_s depends not only on the dissipative forces but also on the gyroscopic forces in the initial system (1.1).

For convenience we shall assume that Eqs (2.3) are written in an orthonormal basis consisting of eigenvectors of the matrix Γ, with the eigenvalues of this matrix γ_j ($j = 1, \dots, n$) arranged in increasing order (recall that they are positive). Then the matrix U has the following appearance

$$U = \|(d_{ij} + g_{ij})\gamma_i\|, \quad D' = \|d_{ij}\|, \quad G' = \|g_{ij}\|; \quad i, j = 1, \dots, n \tag{2.4}$$

For the symmetric part of the matrix U, we obtain

$$U_S = \frac{1}{2}(U + U^T) = \frac{1}{2}\|(\gamma_i + \gamma_j)d_{ij} + (\gamma_i - \gamma_j)g_{ij}\| \tag{2.5}$$

The algorithm for choosing the elements of the matrix G' is associated with stepwise reduction of the matrix (2.5) to block-diagonal form. At the first step the elements of the first row are simplified (or of several rows if γ_1 is a multiple eigenvalue). Let us assume that

$$\gamma_1 = \gamma_2 = \ldots = \gamma_l < \gamma_{l+1}, \quad 1 \le l \le n$$

Setting

$$g_{ij} = d_{ij}(\gamma_i + \gamma_j)/(\gamma_j - \gamma_i); \quad i = 1, \ldots, l; \quad j = l+1, \ldots, n \tag{2.6}$$

we decompose the matrix (2.5) into two square blocks of dimensions l and $n-l$. The first of these blocks is the principal minor of the matrix D' with coefficient γ_1. The second block itself is then simplified by an analogous procedure. In conclusion, depending on the suitable choice of the elements g_{ij}, we obtain $U_s > 0$. By the Thomson–Tate–Chetayev theorem, this implies that system (2.3) is asymptotically stable, and hence the same is true of the initial system (1.1).

This completes the proof of Proposition 1.1.

3. STABILIZATION THROUGH THE USE OF POSITIONAL FORCES

In practice, one frequently encounters situations in which a system must be stabilized using positional non-conservative and gyroscopic forces. For example, in a spacecraft, potential forces arise from the attraction of celestial bodies and cannot be changed. Jet engines create non-conservative forces whose direction relative to the satellite is unchangeable (the so-called "servo forces"), and the spinning rotors create gyroscopic forces.

Proposition 2. The following two properties of a symmetric matrix K' are equivalent:

(1) A skew-symmetric matrix N' exists such that the eigenvalues of the matrix $K' + N'$ are positive and different.

(2) The trace of the matrix K' is positive.

Proof. Since the addition of a skew-symmetric matrix does not alter the trace and the trace is always the sum of the eigenvalues, the second property obviously follows from the first.

To prove the converse implication, we write the matrix K' in terms of an orthonormal basis consisting of its eigenvectors, arranged in order of increasing eigenvalues

$$K' = \operatorname{diag}\{\lambda_i\}, \quad \lambda_j \le \lambda_{j+1}; \quad i = 1, \ldots, n; \quad j = 1, \ldots, n-1$$

We shall assume that $\lambda_1 \le 0$, since otherwise there is nothing to prove. The matrix N' will be constructed by the following procedure of "equalizing" eigenvalues.

Let A be a linear operator in R^n whose matrix relative to some orthonormal basis is upper-triangular

$$A = \|a_{ij}\|, \quad a_{ij} = 0 \quad \text{for} \quad i > j, \quad a_{ii} = \lambda_i; \quad i, j = 1, \ldots, n \tag{3.1}$$

Add to A a skew-symmetric operator with matrix $\Omega = \|\Omega_{ij}\|$ which has only two non-zero elements

$$\Omega_{i,i+1} = -\Omega_{i+1,i} = \omega$$

The number ω is chosen so that the eigenvalues of the operator $A + \Omega$ are

$$\lambda_1, \ldots, \frac{3}{4}\lambda_i + \frac{1}{4}\lambda_{i+1}, \quad \frac{1}{4}\lambda_i + \frac{3}{4}\lambda_{i+1}, \ldots, \lambda_n$$

To that end, as is readily seen, ω must be a root of the equation

$$\omega(\omega + a_{i,i+1}) = 3(\lambda_{i+1} - \lambda_i)^2/16 \tag{3.2}$$

Since the discriminant of the quadratic equation (3.2) is non-negative, it has a real root.

We now change the vectors $\mathbf{e}_i$ and $\mathbf{e}_{i+1}$ in such a way that the matrix of the operator $A + \Omega$ retains its upper-triangular form. The transformation corresponding to this change of basis must conserve the symmetry and skew-symmetry of matrices, that is, it must be orthogonal. This condition can be met by setting

$$\mathbf{e}_i' = \mathbf{e}_i\cos\alpha_i + \mathbf{e}_{i+1}\sin\alpha_i, \quad \mathbf{e}_{i+1}' = -\mathbf{e}_i\sin\alpha_i + \mathbf{e}_{i+1}\cos\alpha_i$$

The angle of rotation α must be chosen so that $((A + \Omega)\mathbf{e}_i', \mathbf{e}_{i+1}') = 0$. To that end, the following equality must hold

$$(a_{i,j+1} + \omega)\operatorname{tg}^2\alpha_i + (\lambda_i - \lambda_{i+1})\operatorname{tg}\alpha_i + \omega = 0 \tag{3.3}$$

The discriminant of the quadratic equation (3.3) is, in view of (3.2),

$$D = (\lambda_{i+1} - \lambda_i)^2/4 \geq 0$$

so that the equation has a real root.

As a result of this "equalizing" procedure, the matrix of the operator has retained its upper-triangular form, while the difference between two adjacent eigenvalues has been halved in value. Successive repetition of the operation (for different i) makes it possible to bring all the eigenvalues of the matrix $K' + N'$ as close to one another as described. Since the sum of these numbers is positive, each of them will be positive. This proves Proposition 2.

Remark. Proposition 2 was stated in [3] without proof. There are examples of the influence of non-conservative forces on stability [2, 4, 5].

Example. The equations of relative motion of a satellite in a circular orbit admit of equilibrium solutions for which its principal central axes of inertia lie along the axes of the orbital system of coordinates [6]. In the neighbourhood of such a solution, the system has the form (1.1), where

$$\begin{aligned} &\mathbf{q} = (\theta, \psi, \varphi)^T, \quad M = \operatorname{diag}\{J_1, J_2, J_3\}, \quad D = G = 0, \quad N = 0 \\ &K = \omega_0^2\operatorname{diag}\{3(J_3 - J_2), J_1 - J_3, 4(J_1 - J_2)\} \end{aligned} \tag{3.4}$$

The coordinates θ, ψ, φ denote the deviations of the angles of nutation, precession, and rotation of the satellite about itself from their equilibrium values $\theta_0 = \pi/2$, $\psi_0 = \pi/2$, $\psi_0 = 0$; J_1, J_2 and J_3 are the principal central moments of inertia, and ω_0 is the angular velocity of orbital motions.

If the inequalities

$$J_1 > J_3 > J_2 \tag{3.5}$$

hold, the satellite's position of relative equilibrium is stable [6]. We shall use the results obtained above to obtain new domains of stability.

We change to Eqs (2.1), using the transformation described at the beginning of Section 2 with the matrix

$$\Lambda = \operatorname{diag}\{J_1^{-1/2}, J_2^{-1/2}, J_3^{-1/2}\}$$

This yields Eqs (2.1), where

$$D' = G' = N' = 0, \quad K' = \omega_0^2\operatorname{diag}\left\{3(u - \upsilon), \frac{1-u}{\upsilon}, 4\frac{1-\upsilon}{u}\right\}; \quad u = \frac{J_3}{J_1}, \quad \upsilon = \frac{J_2}{J_1}$$

The assumption of Proposition 2 is expressed by the inequality

$$\operatorname{tr}(K') = (3(u - \upsilon)u\upsilon + u(1 - u) + 4\upsilon(1 - \upsilon))/(u\upsilon) > 0 \tag{3.6}$$

Obviously, the domain (3.6) is large than (3.5). Let us assume in particular that the body is dynamically symmetric $J_1 = J_2$. Then conditions (3.5) are not satisfied, but inequality (3.6), with $\upsilon = 1$, is valid in the case when $u > 1$. Consequently, an "oblate" satellite whose axis of symmetry points along the tangent to the orbit may be stabilized by the applying non-conservative forces. To that end, we set

$$N' = \|\omega_{ij}\|, \quad \omega_{23} = \varepsilon \ll 1, \quad \omega_{13} = 0$$

and choose the element ω_{12} in such a way that the eigenvalues of the matrix $K' + N'$ are different positive numbers. The characteristic equation has the form

$$\lambda^3 - 2(u-1)\lambda^2 + (\varepsilon^2 + \omega_{12}^2 - 3(u-1)^2)\lambda - 3\varepsilon^2(1-u) = 0 \tag{3.7}$$

If $\varepsilon = 0$, Eq. (3.7) has a zero root, while its other two roots satisfy a quadratic equation with discriminant

$$D = 16(u-1)^2 - 4\omega_{12}^2$$

As is easily verified, the condition for the roots of the quadratic equation to be positive is expressed by the two inequalities

$$\sqrt{3}(u-1) < |\omega_{12}| < 2(u-1) \tag{3.8}$$

If the number ω_{12} is chosen in accordance with (3.8), then, for sufficiently small ε, the roots of Eq. (3.7) will be positive and different. This guarantees stability of the satellite's position of relative equilibrium in the first approximation.

By Proposition 1, if resistance forces with matrix $D > 0$ are present, the equilibrium position can be made asymptotically stable by adding gyroscopic forces. When that is done, it follows from the Thomson–Tate–Chetayev theorem that asymptotic stability will hold in the domain (3.5) for any gyroscopic forces, including zero forces. Outside that domain, the choice of the gyroscopic forces will depend on the dissipative forces. In particular, if the resistance to rotation about any axis passing through the satellite's centre of mass is proportional to the moment of inertia about that axis, then the matrix D' in Eqs (2.1) will be a scalar matrix. After applying the transformation (2.2), the matrix of the velocity-dependent forces remains symmetric and positive. In that case asymptotic stability is achieved without adding gyroscopic forces to the system.

For stabilizing with other resistance laws, one can use the algorithm presented above in the proof of Proposition 1.

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REFERENCES

1. CHETAYEV, N. G., *The Stability of Motion. Papers on Analytical Mechanics*. Izd. Akad. Nauk SSSR, Moscow, 1962.
2. BOLOTIN, V. V., *Non-conservative Problems of the Theory of Elastic Stability*. Fizmatgiz, Moscow, 1961, also Pergamon Press, Oxford, 1963.
3. MÜLLER, P. C., *Stabilität und Matrizen*. Springer, Berlin, 1977.
4. MERKIN, D. R., *Introduction to the Theory of Stability of Motion*. Nauka, Moscow, 1976.
5. ZIEGLER, H., *Principles of Structural Stability*. Blaisdell, Waltham, 1968.
6. BELETSKII, V. V., *The Motion of an Artificial Satellite Relative to its Centre of Mass in a Gravitational Field*. MGU, Moscow, 1975.

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